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Asymmetric wave-stress tensors and wave spin

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Linearized wave-stress tensors derived from Hamilton's variational principle may be asymmetric. If interpreted as momentum fluxes, they would lead to lack of conservation of orbital angular momentum. It is shown that changes in internal angular momentum or spin of waves and torque coupling to external fields can adequately provide conservation of total angular momentum in such cases. Examples are given for acoustic, internal gravity, Rossby and plasma waves.

1. Introduction

One of the standard techniques of both classical and quantum field theory makes use of Hamilton's variational principle to derive conservation equations for an energy–momentum tensor. This tensor is described in terms of the field variables ϕ^i and Lagrangian density of the system. The components of this tensor are identified as energy and momentum densities and fluxes; momentum fluxes are equivalent to the negative of a stress tensor. This identification is made by equating the integral of energy density over the system with the system Hamiltonian; identification of momentum components is not so clear. Both identifications are complicated by the fact that the canonical energy–momentum tensor so derived is unique only to within a divergence.

The variational approach may also be applied to linearized wave theory, using a Lagrangian quadratic in wave perturbations. The Euler–Lagrange equations are then the linearized equations for the system, while the energy–momentum equations describe the behaviour of 'wave energy' and 'wave momentum', which are quantities quadratic in the perturbations. In view of the fact that the quadratic 'wave Lagrangian density' is only part of the total Lagrangian density, as well as the original lack of uniqueness, identification of wave energy and momentum with physical energy and momentum is rather tenuous (cf. Morse & Feshbach 1953; Sturrock 1961, 1962; Bretherton & Garrett 1968; Jones 1971). It may be that the best identifications can be made by *post hoc* considerations of wave energy–momentum tensor components as they are derived for special cases.

We are particularly concerned with spatial components of the wave energy–momentum tensor, which we shall call the wave stress tensor. This stress tensor appears in source terms for wave energy in shearing flow (Garrett 1968; Bretherton & Garrett 1968), as well as in equations for wave momentum. Bretherton (1969) has derived components of this tensor as the vertical flux of horizontal momentum in rotating fluids. The wave-stress tensor forms a part of, though not necessarily all of, the radiation stress tensor defined for acoustic waves (Brillouin 1964) and water waves (Longuet-Higgins & Stewart 1969).

One argument made against the wave-stress tensor as a true stress tensor is that it is frequently not symmetric; if it were then equated to a negative momentum flux it would violate conservation of angular momentum. For example, Sturrock (1961, 1962) finds that wave momentum density equals wave-energy density multiplied by $\mathbf{k}/\omega$, where $\mathbf{k}$ is the wavenumber vector and ω the wave frequency. Wave momentum flux equals wave momentum density multiplied by group velocity. If phase and group velocities are not collinear, and if we interpret wave momentum density as a true momentum density, then the motion of the packet will appear to change the angular momentum of the system.

The argument for symmetry in stress tensors is valid for non-polar materials; it does not hold for polar materials that have an internal angular momentum or spin which can be coupled with orbital angular momentum or moment of linear momentum (McLennan 1966). We shall show that, on averaging over a wave in non-polar materials, one may obtain a spin-like quantity, quadratic in the wave perturbation variables. The conservation equation for this quantity has two source terms. The first is coupling to orbital angular momentum; the second is coupling through torque terms to external fields.

Through consideration of wave spin we can make an asymmetric wave stress tensor consistent with conservation of angular momentum. While the wave changes the orbital angular momentum of the medium, it also exerts a torque on an external body (the earth for internal gravity waves) or field (the zero-order magnetic field for magneto-acoustic and plasma waves) or changes its own internal angular momentum (for obliquely propagating Rossby waves on a β-plane).

In § 2 we review the classical results for linear and angular momentum in polar materials, first by direct arguments and then from the variational point of view. In §§ 3–5 the results are applied to several waves of geophysical and fluid-dynamic interest.

2. Linear and angular momentum

We begin by defining the following scalar and vector and tensor components:

ρ = density,

r^m = m component of the position vector

v^m = m component of velocity (that is, of momentum density divided by mass density)

F^m = m component of external or body force

T^{mn} = stress tensor, force in the m direction per unit area normal to the n axis

R^{mn} = spin or internal angular momentum corresponding to rotation from the m axis to the n axis

G^{mn} = external body torque acting to produce rotation in m–n sense

M^{mnq} = couple stress tensor, producing an m–n torque per unit area normal to the q axis

The stress tensor and couple stress tensor are respectively the negatives of momentum flux and spin flux.

The differential equation for linear momentum is

$$\frac{\partial}{\partial t}(\rho v^m) = \frac{\partial T^{mn}}{\partial x^n} + F^m, \tag{1}$$

while the equation for angular momentum is

$$\frac{\partial}{\partial t}[R^{mn} + \rho(r^m v^n - r^n v^m)] = \frac{\partial}{\partial x^q}[M^{mnp} + (r^m T^{nq} - r^n T^{mq})] + G^{mn} + r^m F^n - r^n F^m. \tag{2}$$

In (1) the stress tensor includes Reynolds as well as material stresses. The left-hand side of (2) describes the rate of change of two kinds of angular momentum; the first is spin or internal angular momentum, the second is the moment of linear momentum. The first term on the right-hand side is the divergence of the total couple stress, consisting of the couple stress tensor and the moment of the linear stress tensor. The final terms are the body torque and the moment of the body force.

Equation (2) may be rewritten with the aid of (1) to give

$$\partial R^{mn}/\partial t = \partial M^{mnq}/\partial x^q + T^{nm} - T^{mn} + G^{mn}. \tag{3}$$

In a non-polar material R^{mn}, M^{mnq} and G^{mn} are all zero. This requires that

$$T^{mn} - T^{nm} = 0, \tag{4}$$

i.e. that the stress tensor is symmetric. This is not true for a polar material, where asymmetry in T^{mn} may be balanced by body torques, time variation in spin density or divergence of the couple stress tensor or spin flux. As the Reynold stress terms $-\rho v^m v^n$, are symmetric, they obviously cancel in (3).

If one has a Lagrangian density L, which is a function of field variables ϕ^i and their derivatives, one can also derive conservation equations for energy and both linear and angular momentum from Hamilton's variational principle. We shall use Greek indices if quantities refer to either time or space, Latin indices if they are spatial co-ordinates and t if time is specifically meant. As we deal with non-relativistic Cartesian co-ordinates, no distinction will be made between contravariant and covariant indices. Repeated indices imply summation and

$$\phi^t_\mu \equiv \partial\phi^i/\partial x^\mu. \tag{5}$$

The variational principle states that

$$\delta \int L d^4x = 0. \tag{6}$$

By choosing specific forms of variation one can derive the Euler–Lagrange equations (Akhiezer & Berestetskii 1965)

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial \phi^i_\mu}\right) - \frac{\partial L}{\partial \phi^i} = 0, \tag{7}$$

and the energy–momentum equations

$$\partial T^{\mu\nu}/\partial x^\nu = 0, \tag{8}$$

where
$$T^{\mu\nu} \equiv \phi^i_\mu(\partial L/\partial \phi^i_\nu) - \delta_{\mu\nu} L. \tag{9}$$

The Euler–Lagrange equations form the governing equations for the system. If T^{tt} is integrated over all space, the integral equals the system Hamiltonian; hence T^{tt} is identified as an energy density. From this conclusion one identifies $-T^{nt}$ as momentum density, T^{tm} as energy flux, $-T^{mn}$ as momentum flux and T^{mn} as a stress tensor. Equation (8) holds only for closed systems, where L has no explicit dependence on x^{μ}.

It is usual to point out that $T^{\mu\nu}$ is not unique, as L is not unique. If $f^{\mu\nu\sigma}$ is any third-order tensor antisymmetric in its last two indices, then

$$\mathring{T}^{\mu\nu} \equiv T^{\mu\nu} + \partial f^{\mu\nu\sigma}/\partial x^{\sigma} \tag{10}$$

also satisfies the conservation equation

$$\partial \mathring{T}^{\mu\nu}/\partial x^{\nu} = 0. \tag{11}$$

This principle is often employed to obtain a symmetric energy–momentum tensor $\mathring{T}^{\mu\nu}$ from the canonical tensor $T^{\mu\nu}$.

If we take variations in the form of infinitesimal rotations, we obtain a conservation equation for angular momentum. Let

$$M^{\mu\nu\sigma} \equiv +x^{\nu}T^{\mu\sigma} - x^{\mu}T^{\nu\sigma} + S^{\mu\nu\sigma}, \tag{12}$$

where

$$S^{\mu\nu\sigma} \equiv I_{\mu\nu ij}\phi^{i}\,\partial L/\partial \phi^{j}_{\sigma}. \tag{13}$$

$I_{\mu\nu ij}$ are the infinitesimal operators of the Lorentz group. For our purposes

$$I_{\mu\nu ij} = \delta_{\mu i}\delta_{\nu j} - \delta_{\mu j}\delta_{\nu i}. \tag{14}$$

Then, from the variational principle, one can show that

$$\partial M^{\mu\nu\sigma}/\partial x^{\sigma} = 0. \tag{15}$$

$M^{\mu\nu\sigma}$ is identified as the angular momentum four-tensor, with $M^{\mu\nu t}$ as density and $M^{\mu\nu m}$ as flux. Total angular momentum is thus conserved.

We shall be concerned only with spatial angular momentum $M^{mn\sigma}$. Total angular momentum density consists of two parts. Orbital angular momentum density

$$x^{n}T^{mt} - x^{m}T^{nt} \tag{16}$$

is the moment of linear momentum density. S^{mnt} is an internal angular momentum density or spin density, which can be identified with R^{mn}.

Equations (8) and (15) would be equivalent to (1) and (2) for a closed system with no external body forces and torques. In the presence of such external forcing, (8) and (15) would have non-zero right-hand sides. These would contribute to an equation for spin conservation.

Let us write an equation for conservation of spin, by differentiating (13) and substituting (9):

$$\partial S^{mn\sigma}/\partial x^{\sigma} = -T^{mn} + T^{nm} + G^{mn}, \tag{17}$$

where

$$G^{mn} \equiv I_{mnij}\left[\phi^{i}\frac{\partial L}{\partial \phi^{j}} + \phi^{i}_{\sigma}\frac{\partial L}{\partial \phi^{j}_{\sigma}} + \phi^{\sigma}_{i}\frac{\partial L}{\partial \phi^{\sigma}_{j}}\right], \tag{18}$$

G^{mn} must be the body torque experienced by an open system, by comparison of (3) and (7). In a closed system G^{mn} is zero, since from (8), (12) and (15) for a closed system

$$\partial S^{mn\sigma}/\partial x^{\sigma} = -T^{mn} + T^{nm}. \tag{19}$$

The variational principle is applied to linearized wave systems by using a Lagrangian density that is quadratic in perturbation quantities. (See Morse & Feshbach (1953) and Jones (1971) for reviews.) The Euler–Lagrange equations are then the linearized perturbation equations for the system. $T^{\mu\nu}$ and $S^{mn\sigma}$ are then quadratic in the perturbations. Interpretation of these quantities as components of total energy and momentum densities and fluxes is now doubly difficult. Not only does the original lack of uniqueness persist, but the quadratic Lagrangian is only a part of the total system Lagrangian and conclusions about wave energy and momentum are as approximate as the linearized perturbation equations. (We shall use the word 'wave' as an adjective, denoting 'quadratic in perturbation variables'.)

In view of these difficulties, *post hoc* identification of the various components is highly desirable. We shall consider our results for a number of waves in the following sections.

3. Stratified fluid waves

We shall begin by using a wave Lagrangian density derived by Hayes (1970) on the basis of an analysis by Eckert (1963):

$$L = \tfrac{1}{2}\rho\xi^r_t\xi^r_t - \tfrac{1}{2}\rho a^2\xi^r_r\xi^s_s - \xi^r_r\xi^s\frac{\partial p}{\partial x^s} - \tfrac{1}{2}\xi^r\xi^s\frac{\partial^2 p}{\partial x^r\partial x^s} - \tfrac{1}{2}\rho\xi^r\xi^s\frac{\partial^2\Phi}{\partial x^r\partial x^s}. \tag{20}$$

This expression is valid for compressible stratified fluids in a gravitational field, and with no mean fluid motion. Thus acoustic and internal gravity waves are described. We define p = mean pressure, ρ = mean density, a = sound speed, $\boldsymbol{\xi}$ = perturbation displacement and Φ = gravitational potential. With no mean motion, and gravity acting in the negative-z direction

$$\partial p/\partial z = -g\rho, \quad g \equiv \partial\Phi/\partial z. \tag{21),(22}$$

Equation (20) may be written as

$$L = \tfrac{1}{2}\rho\xi^r_t\xi^r_t - \tfrac{1}{2}\rho a^2\xi^r_r\xi^s_s + g\rho\,\xi^r_r\xi^s\delta_{sz} + \tfrac{1}{2}g\frac{\partial\rho}{\partial z}\xi^r\xi^s\delta_{rz}\delta_{sz}. \tag{23}$$

Eulerian perturbations ρ' and p' in density and pressure are given by

$$p' = -(\rho a^2\xi^r_r + \xi^r\delta_{rz}\partial p/\partial z) \tag{24}$$

and

$$\rho' = -(\rho\xi^r_r + \xi^r\delta_{rz}\partial\rho/\partial z). \tag{25}$$

The Eulerian–Lagrange equations are then simply

$$\rho\xi^r_{tt} + \partial p'/\partial x^r + g\rho'\delta_{rz} = 0, \tag{26}$$

which are readily seen to be the perturbation equations of motion for this system.

The wave stress tensor is

$$T^{mn} = \xi^n_m p' - \delta_{mn}L. \tag{27}$$

Bretherton (1969) has discussed the physical interpretation of the off-diagonal components of this stress tensor from a particle point of view. Consider a surface element originally normal to the n-axis. Under the action of the wave this surface is moved about and tilted; the pressure on a tilted surface exerts a net force on this surface in the m direction. The wave-stress tensor arises from the correlation of tilt and pressure fluctuations. For an interpretation of the diagonal components see Jones (1971). T^{mn} is not necessarily symmetric.

The wave-spin density is

$$S^{mnt} = I_{mnij}\rho\xi^i\xi^j_t \tag{28a}$$

$$= \rho\xi^m\xi^n_t - \rho\xi^n\xi^m_t \tag{28b}$$

and is simply the cross product of particle displacement and particle velocity. Consider a particle moving in an elliptical orbit. If we average the angular momentum of this particle about some arbitrary origin, we obtain a net angular momentum corresponding to S^{mnt}; it is independent of the position of the origin and thus is spin-like. The particle has no average linear momentum, and hence no mean orbital angular momentum if the latter is taken as the moment of mean linear momentum. It is in this sense that we can speak of the spin of an averaged system which has no inherent polar character.

The wave-spin flux is

$$\begin{aligned} S^{mnq} &= I_{mnij}\xi^i p'\delta_{qj} \\ &= \xi^m p'\delta_{qn} - \xi^n p'\delta_{qm}. \end{aligned} \tag{29}$$

Consider an element of surface originally normal to the n axis. As this is shifted about by the wave motion it may undergo m displacement. If at positive ξ^m the pressure perturbation is positive and vice versa, the integrated force across the element produces a net torque on the fluid beyond it.

Now consider the wave torque

$$G^{mn} = -I_{mnij}g\xi\rho'\delta_{jz}. \tag{30}$$

Specifically

$$G^{xy} = 0, \quad G^{xz} = \xi^x g\rho', \tag{31), (32}$$

but $-g\rho'$ is the buoyant force per unit volume acting in the z direction, and hence G^{xz} is the moment of this force. That is, $g\rho'$ is the net rate at which momentum is lost from a particle of unit volume to the earth, and if fluctuations in this quantity correlate with lateral position, there is a net torque exerted on the earth.

For a uniform periodic wave, the left-hand side of (17) vanishes and we have

$$T^{mn} - T^{nm} = G^{mn}. \tag{33}$$

Thus the change in orbital angular momentum of a fluid induced by a wave packet carrying momentum not collinear with its group velocity is balanced by transfer of angular momentum outside the fluid, in this case to the earth.

4. Waves in rotating systems

We shall drop the effects of buoyancy and include those of an angular velocity $\mathbf{\Omega}$ of the system. This can be done in the usual meteorological approximation including Coriolis forces, but neglecting centrifugal forces (Hayes 1970).

$$L = \tfrac{1}{2}\rho\xi^r_t\xi^r_t - \tfrac{1}{2}\rho a^2\xi^r_r\xi^s_s + \rho\epsilon_{qrs}\xi^q\xi^r_t\Omega^s, \tag{34}$$

where
$$\epsilon_{rqs}\xi^r_t\Omega^s = (\boldsymbol{\xi}_t \times \mathbf{\Omega})^q. \tag{35}$$

The Euler–Lagrange equations are

$$\rho\xi^r_{tt} + \partial p'/\partial x^r - \epsilon_{rqs}2\xi^q_t\Omega^s = 0. \tag{36}$$

The wave-stress tensor and the wave-spin flux are unaltered by rotation, and are given by (27) and (29). The wave-spin density has an added term:

$$S^{mnt} = I_{mnij}\rho[\xi^i\xi^j_t + \epsilon_{kjl}\xi^i\xi\Omega^l]. \tag{37}$$

If the system rotates about the z axis, the added term contributes to S^{xyt} and is

$$\rho\Omega(\xi^x\xi^x + \xi^y\xi^y). \tag{38}$$

A displacement in a rotating system would be viewed as a change in velocity from an inertial system. The added term evidently is the correlation of displacement with this additional velocity perturbation. A related additional term appears in an Eulerian interpretation of momentum flux as the advection of momentum by velocity perturbations (Jones 1971).

The wave torque is

$$G^{mn} = I_{mnij}\,\rho\epsilon_{jkl}(\xi^i\xi^k_t - \xi^i_t\xi^k)\,\Omega^l. \tag{39}$$

Again considering rotation about the z axis,

$$G^{xy} = 0, \tag{40}$$

$$G^{xz} = \rho(\xi^y\xi^z_t - \xi^z\xi^y_t)\,\Omega = S^{yzt}\Omega, \tag{41}$$

$$G^{yz} = -S^{xzt}\Omega. \tag{42}$$

The wave torque is equal to the cross product of wave-spin density and system angular velocity. This is an apparent torque in the same sense that Coriolis forces are apparent forces. Consider a gyroscope rotating about a horizontal axis, while mounted on a plane rotating about a vertical axis. In the absence of any forces, the gyroscope will remain aligned with an inertial axis, but will appear to rotate in the rotating co-ordinate system. There will be an apparent transfer between x–z and y–z angular momenta.

Sound waves propagating with both vertical and horizontal components in such a system have asymmetric stress tensors. On the other hand, sound waves travelling along or transverse to the axis of rotation have

$$G^{mn} = 0, \tag{43}$$

and hence have symmetric stress tensors.

Our equations for wave stress tensors and wave spin tensors are valid in the limit $a \to \infty$, as they are expressed in terms of perturbation pressures and displacements, thus they are valid for non-divergent Rossby waves computed on a β-plane where Ω is a linear function of y. In this case $\xi^z = 0$.

It has been shown (Longuet-Higgins 1964; Buchwalder 1972) that if such waves propagate obliquely in the x, y plane their group and phase velocities are not collinear, and hence that

$$T_{xy} - T_{yx} \neq 0. \tag{44}$$

There are no external torques to balance the changes of orbital angular momentum brought about by such a wave. Instead, we note that S^{xyt} is a function of Ω, and as a packet moves in the y direction, the spin associated with it varies, compensating for the changes in orbital angular momentum. In a time-periodic Rossby wave there will be y divergence of wave spin flux if there is a y component of group velocity.

5. Plasma waves

We now derive the spin relationships for electromagnetic radiation in a cold collisionless plasma with one mobile ion species. We shall assume a mean magnetic field $\mathbf{B}$ and perturbations in the following quantities: $\mathbf{A}$ = vector potential, ϕ = scalar potential and $\boldsymbol{\xi}$ = plasma displacement, from which we can derive perturbations in the electric field intensity

$$\mathbf{E} = -\nabla . \phi - c^{-1}\, \partial \mathbf{A}/\partial t, \tag{45}$$

magnetic field intensity

$$\mathbf{H} = \nabla \times \mathbf{A}, \tag{46}$$

current density

$$\mathbf{J} = \rho e m^{-1} \boldsymbol{\xi}_t, \tag{47}$$

and electric charge density

$$\rho_e = -\rho e m^{-1} \nabla . \boldsymbol{\xi}, \tag{48}$$

where ρ_e = ion mass density, e = ion charge, m = ion mass and c = speed of light.

The variational principle for electromagnetic radiation is normally developed in Lorentz co-ordinates, where $x^4 = ict$ rather than t. We define the four-vectors

$$V^n = A^n, \quad I^n = J^n, \quad V^4 = i\phi, \quad I^4 = ic\rho_e \tag{49}$$

and the four-tensor

$$f^{\mu\nu} = V^\nu_\mu - V^\mu_\nu. \tag{50}$$

The wave Lagrangian density is

$$L = -\frac{1}{16\pi} f^{\mu\nu} f^{\mu\nu} + \frac{i\rho e}{cm}(\xi^n_4 V^n - \xi^m_m V^4) - \frac{\rho c^2}{2} \xi^m_4 \xi^m_4 + \frac{ic\rho e}{2m} \epsilon_{mrs} \xi^m \xi^r_4 B^s. \tag{51}$$

The Euler–Lagrange equations which follow from (51) are

$$\partial f^{\mu\nu} / \partial x^\nu = 4\pi c^{-1} I^\mu, \tag{52}$$

which are Maxwell's equations, and

$$-c^2\rho\xi^m_{44} = -i\rho e\, m^{-1}[(V^m_4 - V^4_m) - c\epsilon_{mqs}\xi^q_4 B^s]. \tag{53}$$

In real space–time co-ordinates, the latter equation becomes

$$\rho\xi^m_{tt} = \rho e m^{-1}[E^m + (\boldsymbol{\xi}_t \times \mathbf{B})^m]. \tag{54}$$

This equation expresses the acceleration of the plasma in response to electric and magnetic fields.

The wave-stress tensor for the entire system is

$$T^{mn} = -\frac{1}{4\pi}(V^\sigma_m V^\sigma_n - V^n_\sigma) - \frac{i\rho e}{m}\xi^n_m V^4 - \delta_{mn} L. \tag{55}$$

We can write the mean of T^{mn} in more familiar terms. One can show that (Morse & Feshbach 1953, p. 329)

$$\begin{aligned} -\langle V^\sigma_m V^\sigma_n - V^n_\sigma\rangle &= -\frac{1}{4\pi}\langle f^{m\sigma} f^{n\sigma}\rangle + \frac{1}{c}\langle V^m I^n\rangle \\ &= \frac{1}{4\pi}\langle E^m E^n + H^m H^n\rangle - \left\langle \frac{i\rho e}{m}\xi^n V^m_{14}\right\rangle + \left\langle \frac{\partial}{\partial x^4}\left(\frac{i\rho e}{m}\xi^n V^m\right)\right\rangle, \end{aligned} \tag{56}$$

where we denote averaging by angular brackets. The last term of (56) becomes zero on averaging over a periodic wave. Similarly,

$$-\left\langle \frac{\rho e}{m}\xi^n_m V^4\right\rangle = \left\langle \frac{\rho e}{m}\xi^n V^4_m\right\rangle \tag{57}$$

and the average of L is zero. Thus

$$\langle T^{mn}\rangle = \frac{1}{4\pi}\langle E^m E^n + H^m H^n\rangle - \left\langle \frac{i\rho e}{m}\xi^n(V^m_{14} - V^4_m)\right\rangle. \tag{58}$$

But

$$-i(V^m_4 - V^4_m) = E^m \tag{59}$$

and

$$\rho e m^{-1}\xi^n = 4\pi P^m, \tag{60}$$

where $\mathbf{P}$ is the electric polarization. As the electric displacement $\mathbf{D}$ is given by

$$\mathbf{D} = \mathbf{E} + \mathbf{P}, \tag{61}$$

we can show that

$$T^{mn} = (4\pi)^{-1}\langle E^m D^n + H^m H^n\rangle. \tag{62}$$

The wave-stress tensor is asymmetric when $\mathbf{D}$ and $\mathbf{E}$ are not collinear, that is to say, when the tensor dielectric coefficient of the system has off-diagonal terms.

The wave spin tensor has components

$$S^{mn4} = I_{mnij}\left[-\rho\xi^i\xi^j_4 + \frac{1}{4\pi}V^i f^{j4} + \frac{i\rho e}{m}\xi^i V^j + \frac{ic\rho e}{m}\epsilon_{qjs}\xi^i\xi^q B^s\right], \tag{63}$$

$$S^{mnq} = I_{mnij}\left[\frac{1}{4\pi}V^i f^{jq} - \frac{i\rho e}{m}\xi^i V^4 \delta_{qj}\right]. \tag{64}$$

The first component of spin density in (63) represents the effect of elliptical particle orbits. The second represents the angular momentum density of electromagnetic radiation *in vacuo*. Circularly polarized radiation has an angular momentum density equal to the radiation energy density divided by wave frequency, and having the same sense of rotation as the electric and magnetic field vectors (Crawford 1965). We have not attempted to interpret the remaining terms, except to note that the final term is analogous to that of a rotating system. The cyclotron frequency $\rho eB/m$ is equivalent to the rotation frequency Ω.

The wave torque is

$$G^{mn} = I_{mnij}\, ic\rho eB^s m^{-1}\, \epsilon_{jks}(\xi^i \xi^k - \xi_4^i \xi^k) \qquad (65)$$
$$= I_{mnij}\,\epsilon_{jks}(\xi^i J^k - \xi^k J^i)\, B^s.$$

If the magnetic field is in the z direction,

$$G^{xy} = 0 \qquad (66)$$

and

$$G^{xz} = (\xi^y J^z - \xi^z J^y)\, B^z. \qquad (67)$$

But $\xi^y J^z - \xi^z J^y$ is simply a component of the magnetic moment created by motion of the charged plasma particles in elliptic orbits, and the wave torque is simply the cross-product of the wave magnetic moment with the mean magnetic field. Thus it is possible for a wave to couple angular momentum to the mean magnetic field, and ultimately back to the sources of this field.

6. Conclusions

We have shown through a number of examples that asymmetric wave-stress tensors are not inconsistent with the interpretation of the wave-stress tensor as the negative of a momentum flux. The quadratic quantities wave-spin density, wave-spin flux, and wave torque generally have simple and straightforward physical interpretations. This is not to say that they can be related to the total momentum budgets of a nonlinear system, only that such budgets may be drawn up consistently with the linearized equations for waves.

Conservation of angular momentum can be related through Noether's theorem to invariance of the Lagrangian density to infinitesimal rotations (Bogoliubov & Shirkov 1959). The examples we have discussed all involve symmetry about an axis defined by the gravitational force, rotation or the alignment of a magnetic field. As in each case our Lagrangian is invariant to rotation about the axis of symmetry, we find that angular momentum about this axis is conserved. This principle should hold for nonlinear as well as linearized systems; we anticipate that conservation of angular momentum about an axis of symmetry also should hold for resonant wave interactions and turbulence.

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